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On the Numerical Solution of a Heat Equation Associated with a Thermal Print-Head

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Alternating direction implicit and locally one dimensional methods are considered for the solution of a heat equation with constant coefficients, defining the heat flow in a thermal print-head. One of the boundary conditions is unusual in that it constitutes the solution of a heat equation in two space variables in which a heat source is discontinuous in the space and time variables. Several numerical examples are considered to study the behaviour of the difference schemes as a result of these discontinuities.

I. INTRODUCTION

The rapid development of modern technology and engineering has brought with it physical problems of a very complex nature. These physical problems in turn give rise to mathematical models of such complexity that analytical means of deriving solutions are nearly always impossible. Hence, the numerical techniques devised over recent years have come to play an important role in determining solutions to the physical problems that arise. Moreover, in some cases, numerical procedures have been devised for mathematical models for which existence and uniqueness proofs have been lacking; for example, the general Navier Stokes equations (see, for example, Scala and Gordon [23] and Thommen [24]). A similar situation exists with regard to the problem which is to be considered in this paper. There have been (to the author's knowledge) no existence or uniqueness proofs in the literature for the mathematical model we shall consider.

The physical problem consists of determining the flow of heat in a thermal print head subject to a discontinuous heat source which is generated in a thin film deposited on the surface of a glass substrate. The description of the problem and its associated mathematical model are given in Section II. In Section III the difference schemes for the solution of the heat equation in the glass substrate are discussed. In Section IV the difference schemes for the solution of the heat equation in the thin film are considered. Section V presents the results of some numerical experiments using the methods outlined in Sections III and IV. The paper is concluded in Section VI with a discussion and explanation of the numerical results.

II. The Physical Problem and Its Associated Mathematical Formulation

The thermal print head is a matrix of thin film heating resistors deposited on a glass substrate. The surface of the glass and resistors is covered with a thin film of a good heat conducting material. Details of the matrix are given, from a top-view, in Fig. 1. Fig. 2 describes a single element of the matrix from a side-view.

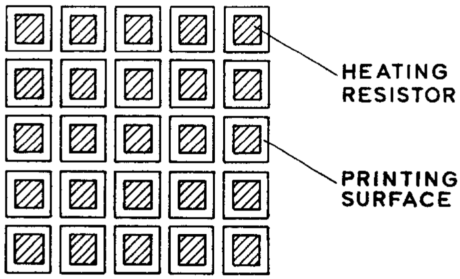

FIG. 1. A 5 × 5 matrix thermal print head (top view).

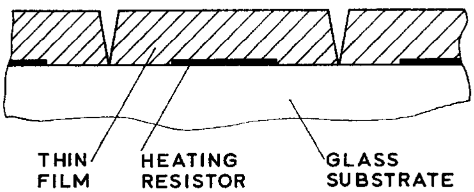

FIG. 2. A single element of the matrix (side view).

When electrical current is passed through a heating resistor, it heats the thin film matrix element directly above it. By passing heat-sensitive paper over the matrix at a prescribed rate, characters can be generated on the paper by passing current through different combinations of the heat resistors. The heat-sensitive paper has a threshold temperature; above it, a chemical reaction in the paper takes place and the characters are formed, and, below it, the paper is left "clean." When one character has been completed, the current is switched off and the heat flows through the glass substrate. With a rapid switching of the heat sources, a build up of heat will occur in the substrate, unless sufficient time is allowed for the heat to dissipate. It is this problem of heat generation and heat flow that we wish to consider in the present paper.

We shall consider the heat flow in a single element of the matrix; that this is not restrictive will be discussed in Section VI. The mathematical model describing the physical problem was originally formulated by Chen [4]. This formulation assumes the thickness of the thin film to be so small in comparison with that of the glass substrate, and the thermal conductivity coefficient for the thin film to be sufficiently high so as to make the temperature gradient in the vertical direction negligible. If we represent the vertical direction in a cartesian co-ordinate system (x, y, z, t), this effectively means that the thin film has no dimension in the z coordinate other than to give it thermal capacity due to a thickness Δ, say.

The heat equation governing the heat distribution in the thin film is then ([4])

$$\frac{\partial u}{\partial t} = \frac{K}{\rho c}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{1}{\rho c}[q\{1 - H(t - t_0)\} \times \{H(x - a) - H(x - 2b)\}\{H(y - a) - H(y - 2b)\}] - \frac{h_0}{\rho c \Delta}(u - u_\infty), \tag{1}$$

where $H(\theta)$ is the Heaviside function defined by

$$H(\theta) = \begin{cases} 0 & \theta < 0, \\ 1 & \theta > 0, \end{cases}$$

$u = u(x, y, 0, t)$ denotes the temperature in degrees centigrade at a point $(x, y, 0, t)$ in the thin film, K is the thermal conductivity, ρ the density, c the specific heat of the thin film, u_∞ is the ambient temperature. q is the heat generated in watts per unit volume, h_0 is the convective heat transfer coefficient between the thin film and air (see, for example, Carslaw and Jaeger [3], p. 15), t_0 (>0) is the value of the time at which the power is switched off. (The form of the heat source term arising from several switchings of the heat source can be expressed in terms of Heaviside functions; the exact form is obvious and will be omitted.) The heating resistor is defined as the square $\{(a, a) \leqslant (x, y) \leqslant (2b, 2b)\}$, where $a + 2b = \ell$, the side length of the print head.

The initial condition $u(x, y, 0, 0) = f(x, y, 0)$, $0 \leqslant x, y \leqslant \ell$ and the boundary conditions $\partial u/\partial n = 0$, $x = 0, \ell$; $0 \leqslant y \leqslant \ell$, $y = 0, \ell$; $0 \leqslant x \leqslant l$, are given for Eq. (1), where n is the outward drawn normal to the edges of the thin film and f is a continuous function. We assume continuity of initial and boundary conditions.

The region in which the solution is required is defined by

$$\bar{R} = R \times [0 < t \leqslant T],$$

where $R = \{(x, y, z); 0 < x, y, z < \ell\}$,
and we denote the boundary of $\bar{R}$ by $\partial\bar{R}$ so that the solution of Eq. (1), with initial conditions and boundary conditions, constitutes a boundary condition on $\partial\bar{R}_{z=0}$ for the total print head.

The equation governing the temperature distribution u in the glass substrate is

$$\frac{\partial u}{\partial t} = \frac{K_1}{\rho_1 c_1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{3}$$

subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z), \quad 0 \leqslant x, y, z \leqslant \ell \tag{4}$$

and the boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \bar{R}_{x=0,\ell}, \qquad \text{and} \qquad \partial \bar{R}_{y=0,\ell} \tag{5}$$

$u(x, y, \ell, t) = g(x, y, \ell, t)$, and $u(x, y, 0, t)$ is the solution of Eqs. (1) and (2) on $\partial \bar{R}_{z=\ell,0}$, respectively. K_1, ρ_1, c_1 are the conductivity coefficient, density, and specific heat, respectively of the glass. g is a given continuous function. The print head has been assumed a cube for programming convenience; the results are true for any rectangular shaped print head.

Thus, our problem constitutes a linear second initial-boundary value problem with constant coefficients in three space variables with one of the boundary conditions given by the solution of a semilinear second initial-boundary value problem of lesser degree which contains a discontinuous heat source. (For definitions of these terms, see Friedman [12]).

III. The Difference Schemes in Three Space Variables

In this section we shall consider difference methods for the solution of the heat Eq. (3), together with the initial and boundary conditions (4) and (5). The difference schemes to be considered will be of two distinct types, namely, Alternating Direction Implicit (A.D.I.) methods of the kind first introduced by Peaceman and Rachford [20] in 1955, alternatively formulated by Douglas and Rachford [8] in 1956, and the Locally One Dimensional (L.O.D.) methods introduced by Yanenko [25] and discussed by D'Yakonov [5], Samarskii [22], and more recently by Fryazinov [12]. (See also the references contained in these papers.)

Before discussing these schemes, it will be convenient to present the notation which we shall use. A rectilinear grid is superimposed on the region of computation $\bar{R}$, where the mesh spacings in the space directions are taken equal; namely,

$$\Delta_x = \Delta_y = \Delta_z = h,$$

and the mesh spacing in the time direction Δ_t is denoted by τ. We shall assume the mesh ratio τ/h^2 to be constant and equal to r. We denote by $u_m = u_{ijk}^m$, the value of the unknown function u at the point $(ih, jh, kh, m\tau) = (x, y, z, t)$ for

$$i, j, k = 0, 1, \ldots, N; \quad Nh = \ell, \quad \text{and} \quad m = 0, 1, 2, \ldots .$$

The difference operators δ_x, δ_y and δ_z are the usual central difference operators, where

$$\delta_x u_{ijk}^m = u_{i+1/2,j,k}^m - u_{i-1/2,j,k}^m ,$$

and similar expressions for δ_y and δ_z.

III. (a) The L.O.D. Scheme of Accuracy $h^2 + \tau^2$

It may be easily verified that an order $(h^2 + \tau^2)$ accurate approximation to Eq. (3) is given by

$$(I + \mu W)(I + \mu V)(I + \mu U)\, \mathbf{u}_{m+1} = (I - \mu W)(I - \mu V)(I - \mu U)\, \mathbf{u}_m + \mathbf{g}_{m+1,m} , \tag{6}$$

where $\mathbf{u}_{m+1}$ and $\mathbf{u}_m$ are the vectors of the unknown temperatures, for

$$i, j, k = 0, 1, \ldots, N. \qquad \mu = \frac{r}{2} \frac{K_1}{\rho_1 c_1} .$$

The vector $\mathbf{g}_{m+1,m}$ contains the values of the function u occurring as a result of the difference scheme, in the local sense, being applied to points adjacent to boundary points; that is, $\mathbf{g}_{m+1,m}$ contains boundary values from $m\tau$ and $(m + 1)\,\tau$. I is the unit matrix of order $(N + 1)^3$. The matrices W, V, and U arise from the representation of the boundary conditions in difference form and the difference operators δ_z^2, δ_y^2, and δ_x^2, respectively, which appear in the local representation of the partial differential equation, namely

$$(1 - \mu\delta_z^2)(1 - \mu\delta_y^2)(1 - \mu\delta_x^2)\, u_{m+1} = (1 + \mu\delta_z^2)(1 + \mu\delta_y^2)(1 + \mu\delta_x^2)\, u_m .$$

The particular forms of W, V, and U may be easily shown to be

$$W = W_{N+1} \otimes I_{N+1} \otimes I_{N+1} ,$$

$$V = I_{N+1} \otimes V_{N+1} \otimes I_{N+1} ,$$

$$U = I_{N+1} \otimes I_{N+1} \otimes U_{N+1} ,$$

where I_{N+1} is the unit matrix of order $N + 1$ and the matrices W_{N+1}, V_{N+1}, and U_{N+1} are of order $N + 1$ and defined by

$$W_{N+1} = \begin{bmatrix} 2 & 0 & & & & & \\ -1 & 2 & -1 & & & 0 & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & 0 & & & -1 & 2 & -1 \\ & & & & & 0 & 2 \end{bmatrix}$$

$$V_{N+1} = U_{N+1} = \begin{bmatrix} 2 & -2 & & & & & \\ -1 & 2 & -1 & & & 0 & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & 0 & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix}$$

and $\otimes$ is the usual tensor product (see, for example, Halmos [17]). It will be noticed that the particular form of the matrices comes about from the particular form of the boundary conditions. The fact that V_{N+1} equals U_{N+1} plays a significant part in the analysis which will be presented.

In order to solve Eq. (6) for the unknown vector $\mathbf{u}_{m+1}$, the equation has to be split into equations of simpler form. The first splitting of Eq. (6) we shall consider will be the L.O.D. scheme. Consider the factorization of Eq. (6) given by

$$(I + \mu U)\,\mathbf{v}_{m+1}^{(*)} = (I - \mu U)\,\mathbf{v}_m + \mathbf{b}_1, \tag{7}$$

$$(I + \mu V)\,\mathbf{v}_{m+1}^{(**)} = (I - \mu V)\,\mathbf{v}_{m+1}^{(*)} + \mathbf{b}_2, \tag{8}$$

$$(I + \mu W)\,\mathbf{v}_{m+1} = (I - \mu W)\,\mathbf{v}_{m+1}^{(**)} + \mathbf{b}_3, \tag{9}$$

where the symbols (*) and (**) indicate intermediate solutions. The vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ contain contributions (as yet undetermined) from the boundary value elements contained in the vector $\mathbf{g}_{m+1,m}$. The relationship between the vectors $\mathbf{v}$ and the solution $\mathbf{u}$ will become apparent after a little further analysis. This analysis is an obvious extension to three space variables of results presented by Gourlay and Mitchell [15].

On elimination of the two intermediate levels (*) and (**) from equations (7), (8), and (9), we obtain

$$
\begin{aligned}
\mathbf{v}_{m+1}^{(*)} &= (I + \mu U)^{-1}(I - \mu U)\,\mathbf{v}_m + (I + \mu U)^{-1}\mathbf{b}_1, \\
\mathbf{v}_{m+1}^{(**)} &= (I + \mu V)^{-1}(I - \mu V)\,\mathbf{v}_{m+1}^{(*)} + (I + \mu V)^{-1}\mathbf{b}_2, \\
&= (I + \mu V)^{-1}(I - \mu V)(I + \mu U)^{-1}(I - \mu U)\,\mathbf{v}_m \\
&\quad + (I + \mu V)^{-1}[\mathbf{b}_2 + (I - \mu V)(I + \mu U)^{-1}\mathbf{b}_1], \\
(I + \mu W)\,\mathbf{v}_{m+1} &= (I - \mu W)\,\mathbf{v}_{m+1}^{(**)} + \mathbf{b}_3, \\
&= (I - \mu W)(I + \mu V)^{-1}(I - \mu V)(I + \mu U)^{-1}(I - \mu U)\,\mathbf{v}_m \\
&\quad + \mathbf{b}_3 + (I - \mu W)(I + \mu V)^{-1}[\mathbf{b}_2 + (I - \mu V)(I + \mu U)^{-1}\mathbf{b}_1].
\end{aligned}
\tag{10}
$$

Putting $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{0}$, Eq. (10) becomes

$$(I + \mu W)\,\mathbf{v}_{m+1} = (I - \mu W)(I + \mu V)^{-1}\,(I - \mu V)(I + \mu U)\,\mathbf{v}_m + \mathbf{b}_3. \tag{11}$$

Since $U_{N+1} = V_{N+1}$, U and V commute and, hence, $(I + \mu V)^{-1}$ and $(I - \mu U)$ commute. Thus, Eq. (11) can be written in the form

$$(I + \mu W)\,\mathbf{v}_{m+1} = (I - \mu W)(I - \mu V)(I - \mu U)(I + \mu V)^{-1}\,(I + \mu U)^{-1}\,\mathbf{v}_m + \mathbf{b}_3. \tag{12}$$

On introducing the transformation

$$(I + \mu V)^{-1}\,(I + \mu U)^{-1}\,\mathbf{v}_m = \mathbf{u}_m,$$

Eq. (12) can then be written as

$$(I + \mu W)(I + \mu V)(I + \mu U)\,\mathbf{u}_{m+1} = (I - \mu W)(I - \mu V)(I - \mu U)\,\mathbf{u}_m + \mathbf{b}_3, \tag{13}$$

where we have once again used the commutativity property mentioned earlier. We see that Eqs. (6) and (13) are equivalent if $\mathbf{b}_3$ is chosen equal to $\mathbf{g}_{m+1,m}$. Hence, an L.O.D. splitting which is $O(h^2 + \tau^2)$ accurate is

$$
\begin{aligned}
(I + \mu U)\,\mathbf{v}_{m+1}^{(*)} &= (I - \mu U)\,\mathbf{v}_m, \\
(I + \mu V)\,\mathbf{v}_{m+1}^{(**)} &= (I - \mu V)\,\mathbf{v}_{m+1}^{(*)}, \\
(I + \mu W)\,\mathbf{v}_{m+1} &= (I - \mu W)\,\mathbf{v}_{m+1}^{(**)} + \mathbf{g}_{m+1,m},
\end{aligned}
\tag{14}
$$

together with the transformation

$$(I + \mu V)^{-1}(I + \mu U)^{-1}\mathbf{v}_m = \mathbf{u}_m . \tag{15}$$

This approximation (6) split in the form (14) and (15) is to be preferred, on the grounds of computational efficiency, to the $O(h^2 + \tau^2)$ scheme

$$(I + \mu U)(I + \mu V)(I + \mu W)\,\mathbf{u}_{m+1} = (I - \mu U)(I - \mu V)(I - \mu W)\,\mathbf{u}_m + \mathbf{g}_{m+1,m} , \tag{16}$$

split in the form

$$\begin{aligned}(I + \mu W)\,\mathbf{v}_{m+1}^{(*)} &= (I - \mu W)\,\mathbf{v}_m ,\\ (I + \mu V)\,\mathbf{v}_{m+1}^{(**)} &= (I - \mu V)\,\mathbf{v}_{m+1}^{(*)} ,\\ (I + \mu U)\,\mathbf{v}_{m+1} &= (I - \mu U)\,\mathbf{v}_{m+1}^{(**)} + (I + \mu V)^{-1}\mathbf{g}_{m+1,m} ,\end{aligned} \tag{17}$$

together with the transformation

$$\mathbf{u}_m = (I + \mu W)^{-1}\,\mathbf{v}_m . \tag{18}$$

[That {(17), (18)} is equivalent to (16) can be proved by a similar analysis used to prove the equivalence of {(14), (15)} and (6).] Equation (17) requires the inversion of one operator only at the transformation stage of the calculation, whereas Eq. (14) requires the inversion of two operators in (15). However, the final equation of (17) requires the inversion of the operator $(I + \mu V)$ each time it is used, whereas Eq. (14) does not. The more efficient scheme is determined by the fact that the transformations need to be carried out only at the print-out stage of the computation [15], and assuming one does not print out the solution every time step, the number of inversions carried out in the first splitting can be considerably less than that in the second one.

III. (b) THE A.D.I. SCHEME

In [9], Fairweather and Mitchell introduced A.D.I. methods which were $O(h^2 + \tau^2)$ accurate for a heat equation with constant coefficients in three space variables. This scheme, however, suffered from the severe stability condition $r \leqslant 11/12$. In [11], Fairweather, Gourlay, and Mitchell introduced a scheme which was again $O(h^2 + \tau^2)$ accurate but which was unconditionally stable ($r > 0$). Various splittings for this method have been proposed. We prefer, in this problem, to use the D'Yakonov splitting [6] owing to its simplicity, although computationally it is the least efficient of the available A.D.I. splittings; see [7] and [16]. (The justification for having chosen this particular factorization will become clear in the discussion in Section VI.)

The particular factorization referred to is

$$\begin{aligned}(I + \mu W)\,\mathbf{u}^{(*)}_{m+1} &= (I - \mu W)(I - \mu V)(I - \mu U)\,\mathbf{u}_m + \mathbf{g}_{m+1,m}\,, \\ (I + \mu V)\,\mathbf{u}^{(**)}_{m+1} &= \mathbf{u}^{(*)}_{m+1}\,, \\ (I + \mu U)\,\mathbf{u}_{m+1} &= \mathbf{u}^{(**)}_{m+1},\end{aligned} \tag{21}$$

where $\mu = K_1 r/(2\rho_1 c_1)$.

All the schemes discussed in Section III are unconditionally stable.

III. (c) Incorporation of Boundary Conditions

The effects of the incorrect incorporation of boundary conditions at the intermediate levels when the boundary conditions are time dependent, has long been well-known (see, for example [6], [10]). The boundary condition at $z = 0$ in the current problem is certainly time dependent. It would appear, therefore, that we have to take care in the method of incorporating the intermediate boundary conditions. Fairweather and Mitchell [10] and Gourlay and Mitchell [14] presented techniques for incorporating intermediate boundary conditions for the local difference schemes. However, by considering the difference method globally, as we have done here, the difficulty is automatically taken care of. That is, the choice of the vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, in terms of the vector $\mathbf{g}_{m+1,m}$, has ensured no loss of accuracy due to incorporation of boundary conditions since the elimination of the intermediate levels yields the original unsplit scheme (6).

IV. The Difference Schemes in Two Space Varialbes

We will now consider finite difference schemes for the solution of Eq. (1) and the associated initial and boundary conditions (2). Owing to the complicated form of the source term, we prefer to consider Eq. (1) in the form

$$\frac{\partial u}{\partial t} = \frac{K}{\rho c}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + S(u, x, y, t), \tag{22}$$

where S can be *any* general source term. This will enable us to generate methods which will cover nonlinear source terms other than the specific one defined in Eq. (1), yet, which will generate, simply, the solution for a source term which is linear in u. Since S is a function of the unknown variable, u, we will require a predictor formula in order to attain $O(h^2 + \tau^2)$ accurate methods. A previous method proposed for Eq. (22) has been given by Albrecht [1]. There, an iteration process was introduced to determine an approximation to u which was used in the source term. A Peaceman Rachford factorization was used and no mention of intermediate boundary conditions was made. We propose to use a scheme which is computationally less time-consuming. Throughout, we will use the predictor formula

$$u_{m+1}^{(1)} = u_m + br(K/\rho c(\delta_x^2 u_m + \delta_y^2 u_m) + h^2 S_m), \tag{23}$$

where b is subsequently to be determined, and where $u_m = u_{ij0}^m$.

IV. (a) THE L.O.D. SCHEME IN TWO SPACE VARIABLES

We propose the L.O.D. scheme written locally as

$$\begin{aligned}(1 - \alpha\delta_x^2)\, u_{m+1}^{(*)} &= (1 + \beta\delta_x^2)\, u_m\,,\\ (1 - \alpha\delta_y^2)\, u_{m+1} &= (1 + \beta\delta_y^2)\, u_{m+1}^{(*)} + \gamma \bar{S}_{m+1/2}^{(1)}\,,\end{aligned} \tag{24}$$

together with Eq. (23). $\bar{S}_{m+1/2}^{(1)} = S[(1 - \alpha\delta x^2)\, u_{m+1}^{(1)}, ih, jh, (m + \frac{1}{2})\,\tau]$.

Eliminating the (*) and (1) levels in (24) and expanding the resulting expression using Taylor series in terms of u_m and its derivatives, and finally using Eq. (22), it may be shown that

$$\gamma = \tau, \qquad \alpha = \beta = Kr/(2\rho c), \qquad b = \tfrac{1}{2},$$

in order that Eq. {(23), (24)} be $O(h^2 + \tau^2)$ accurate, where we have used the fact that the difference operators δx^2, δy^2 commute. Theoretically, the solution is now possible when the usual difference form of the derivative boundary conditions are included. However, in practice, we are faced with the problem that $\bar{S}_{m+1/2}^{(1)}$ will depend upon the values of $u_{m+1}^{(1)}$ outside the region of computation. This problem could be resolved using a 'boundary-inversion' technique similar to that outlined in [16] for hyperbolic systems; but this complicates matters. Since the problem we consider has the function S defined to be nonzero between the stipulated limits [see Eq. (1)], we could solve the problem by noting that the values of the function $u_{m+1}^{(1)}$ required outside the region in which S was nonzero have been calculated, and in this particular case there is no problem. However, as we stated at the beginning of this section, we require to be as general as possible in dealing with the source term S and will, therefore, attempt a method of solution which does not have any draw backs outlined above.

The basis of this approach is that, independent of the region of definition of S, the matrices that arise in the global representation of (24); namely,

$$\tilde{V} = V_{N+1} \otimes I_{N+1},$$
$$\tilde{U} = I_{N+1} \otimes U_{N+1},$$

where I_{N+1}, V_{N+1} and U_{N+1} are defined in Section III.(a), commute. That is, $I + \alpha\tilde{U}$ and $I - \beta\tilde{V}$ commute. We can, therefore, apply the more usual form of elimination of the intermediate level as follows:

We represent the L.O.D. scheme globally as

$$(I + \alpha\tilde{U})\, \mathbf{u}^{(*)}_{m+1} = (I - \beta\tilde{U})\, \mathbf{u}_m,$$
$$(I + \alpha\tilde{V})\, \mathbf{u}_{m+1} = (I - \beta V)\, \mathbf{u}^{(*)}_{m+1} + \gamma \mathbf{S}^{(1)}_{m+1/2},$$

where

$$\mathbf{S}^{(1)}_{m+1/2} = S(\mathbf{u}^{(1)}_{m+1}, ih, jh, (m + \tfrac{1}{2})\tau); \qquad \alpha = \beta = Kr/(2\rho c) \quad \text{and } \gamma$$

is as yet undetermined. If we eliminate the intermediate (*) level, using the fact that $\tilde{U}$ and $\tilde{V}$ commute, we obtain

$$(I + \alpha\tilde{U})(I + \alpha\tilde{V})\, \mathbf{u}_{m+1} = (I - \beta\tilde{U})(I - \beta\tilde{V})\, \mathbf{u}_m + (I + \alpha U)\, \gamma \mathbf{S}^{(1)}_{m+1/2}. \qquad (26)$$

If (26) is now expanded in terms of Taylor series, it may easily be seen that in order that (26) be accurate to order $(h^2 + \tau^2)$,

$$\gamma = \tau(I + \alpha\tilde{U})^{-1}.$$

Hence, the scheme we propose is

$$(I + \alpha\tilde{U})\, \mathbf{u}^{(*)}_{m+1} = (I - \beta\tilde{U})\, \mathbf{u}_m,$$
$$(I + \alpha\tilde{V})\, \mathbf{u}_{m+1} = (I - \beta\tilde{V})\, \mathbf{u}^{(*)}_{m+1} + \tau \mathbf{w}^{(1)}_{m+1/2},$$

where

$$\mathbf{w}^{(1)}_{m+1/2} = (I + \alpha\tilde{U})^{-1}\, \mathbf{S}^{(1)}_{m+1/2}.$$

IV. (b) A.D.I. Schemes in 2 Space Dimensions

These schemes are the Fairweather-Gourlay-Mitchell scheme modified to cope with the source term S. The $O(h^2 + \tau^2)$ scheme is readily shown to be

$$\begin{aligned} (I + \alpha\tilde{V})\, \mathbf{u}^{(*)}_{m+1} &= (I - \beta\tilde{V})(I - \beta\tilde{U})\, \mathbf{u}_m + \tau \mathbf{S}^{(1)}_{m+1/2}, \\ (I + \alpha\tilde{U})\, \mathbf{u}_{m+1} &= \mathbf{u}^{(*)}_{m+1}, \end{aligned} \qquad (29)$$

where $\alpha = \beta = Kr/(2\rho c)$ yields the $O(h^2 + \tau^2)$ A.D.I. scheme.

IV. (c) Boundary Conditions

Since the normal boundary conditions are given to be zero for the two dimensional problem, no inhomogeneous terms are present. Hence, if we were to introduce a vector $\mathbf{g}_{m+1,m}$ similar to Section III, all its entries would be zero. For this reason we have omitted such a vector. If the need arose, it could be incorporated in an obvious manner.

We finally note in this section that the difference schemes discussed have been unconditionally stable ($r > 0$) using the usual stability analysis for the difference schemes in the local sense (see, for example, Richtmyer and Morton [21]). This is also true in the global sense when the particular form of the matrices U, V, and W are considered as a result of the introduction of the mixed Neumann and Dirichlet [illegible]

V. Numerical Results

A series of numerical experiments was carried out in order to test the behaviour of the difference schemes discussed in Sections III and IV. It has been reported by Chen [4] that the incorporation of a source term which contained a discontinuity in time could lead to the propagation of large errors *after* the heat source was switched off. In the report [4], the author claimed a necessity to reduce the mesh size in the region of the switch-off in order to prevent such error growth. We, however, feel that such errors propagated from an incorrect introduction of boundary data and that by incorporating the intermediate boundary data in a manner outlined in Sections III and IV, the difference schemes do not lose accuracy or suffer from the reported error growth despite the switch-off of a large heat source or the frequent switching of the current. In [15] Gourlay and Mitchell showed the equivalence of the L.O.D. schemes and the A.D.I. schemes in two space variables under the transformations mentioned above. The use of the two types of methods is, therefore, two-fold. First, the two independent schemes give a check on the numerical results obtained in the case where no theoretical solution is known. Second, the equivalence of the two types of schemes can be checked on a physical problem, and assuming the equivalence to be borne out, the choice of method reduces to one of efficiency and ease of use.

V. (a) A Problem with a Constant Heat Source ($\tau_0 = \infty$) with a Known Theoretical Solution

To test the schemes under ideal (model) conditions, we constructed a problem for which we were able to determine a theoretical solution. Under these conditions, we could check the equivalence of the schemes and also determines the accuracy of the methods. A solution to Eq. (3) with the associated initial and boundary conditions was found to be

$$u = \left(e^{\gamma t} \cos \frac{\pi x}{\ell} \cos \frac{\pi y}{\ell} + \frac{\Delta}{h_0} q\right)\left(1 - \frac{z}{\ell}\right) + e^{\lambda t} \cos \frac{\pi x}{\ell} \cos \frac{\pi y}{\ell} \sin \frac{\pi z}{\ell}, \qquad (30)$$

where

$$\gamma = -\frac{2\pi^2 K_1}{\ell^2 \rho_1 c_1}, \qquad \lambda = -\frac{3\pi^2 K_1}{\ell^2 \rho_1 c_1}, \qquad \Delta = \frac{h_0 \ell^2}{2\pi^2 \left(\dfrac{K_1}{\rho_1 c_1} - \dfrac{K}{\rho c}\right) \rho c},$$

and where the initial condition associated with this particular solution is obtained by putting $t = 0$ in the expression (30). u_∞ was taken equal to zero, and the boundary condition $u(x, y, \ell, t) = g(x, y, \ell, t)$ taken equal to 0 for all time. For $z = 0$ in (30), the solution in the thin film is obtained. We have taken $a = 0$ and $2b = \ell$ in Eq. (1). (The analysis for the derivation of the solution (30) has been omitted in order to keep the analysis to a minimum.)

The results of computations using

(a) The A.D.I. schemes in two dimensions and three dimensions and

(b) The L.O.D. scheme in two dimensions and three dimensions are quoted in Tables I and II.

TABLE I

Maximum error at 100 time steps in the thin film ($h = 0.1$)

r	A.D.I. scheme	L.O.D. scheme
0.1	1.88×10^{-4}	1.88×10^{-4}
0.3	5.3×10^{-4}	5.3×10^{-4}
0.6	9.6×10^{-4}	9.6×10^{-4}
1.0	1.42×10^{-3}	1.42×10^{-3}

TABLE II

Maximum error at 100 time steps in the glass substrate ($h = 0.1$)

r	A.D.I. scheme	L.O.D. scheme
0.1	6.14×10^{-4}	6.14×10^{-4}
0.3	1.69×10^{-3}	1.69×10^{-3}
0.6	2.98×10^{-3}	2.98×10^{-3}
1.0	4.22×10^{-3}	4.22×10^{-3}

The errors in Tables I and II correspond to the points at which the solution (30) took on its maximum values in the thin film and glass, respectively. The equivalence of the L.O.D. and A.D.I. schemes is evident. Henceforth, we will quote only those results obtained from the A.D.I. scheme, although the results were obtained for the L.O.D. method as well.

V. (b) A Problem with a Discontinuous Heat Source

Having tested the programmes for the finite difference schemes in which the differential equations were well behaved, a problem was considered with a discontinuous heat source, namely, the heat source initially on, was switched off after t_0 sec had elapsed. A solution covering the whole print head was not found. However, a solution in the thin film was derived for all time. The initial conditions for the whole block were then given so that no inconsistency arose between the glass and thin film. The solution for Eq. (1) with $u_\infty = 0$ and $a = 0$, $2b = \ell$ was

$$u = \frac{\Delta}{h_0} q(1 - e^{\nu(t-t_0)})[1 - H(t - t_0)] + \cos\frac{\pi x}{\ell}\cos\frac{\pi y}{\ell}\, e^{[\nu+(2\sigma\pi^2/\ell^2)]t}, \quad (31)$$

where

$$\nu = -\frac{h_0}{\rho c\Delta} \quad \text{and} \quad \sigma = \frac{K}{\rho c}.$$

The initial condition for Eq. (31) is

$$u\,|_{t=0} = \frac{\Delta}{h_0} q(1 - e^{-\nu t_0}) + \cos\frac{\pi x}{\ell}\cos\frac{\pi y}{\ell}, \quad (32)$$

where we have assumed $t_0 > 0$. The details of the derivation will be omitted but a check is obtained by differentiating Eq. (31) and using the fact that

$$\delta(t - t_0)\, e^{\nu(t-t_0)} = 0$$

(see Jones [18]), where δ is the Dirac delta function.

The initial condition which was chosen consistent with Eq. (32) [the equation reduced to (32) on substitution of $z = 0$] was given by

$$u\,|_{t=0}^{\text{glass}} = \left\{\frac{\Delta}{h_0} q(1 - e^{-\nu t_0}) + \cos\frac{\pi x}{\ell}\cos\frac{\pi y}{\ell}\right\}\left\{1 - \frac{z}{\ell}\right\}$$
$$+ \cos\frac{\pi x}{\ell}\cos\frac{\pi y}{\ell}\sin\frac{\pi z}{\ell}. \quad (33)$$

This initial condition in the glass allows a genuine growth (decay) in the glass owing to the term $\cos(\pi x/\ell) \cos(\pi y/\ell) \sin(\pi z/\ell)$. The boundary condition $u(x, y, \ell, t)$ was chosen to be zero for all time. The errors in the thin film for various values of the mesh ratio and heat source q are given in Table III for $t = t_0 - \tau$, $t = t_0 + \tau$ and $t = 100\tau$. t_0 was chosen to be 41 time steps.

TABLE III

Maximum Errors in the Thin Film ($h = 0.1$)

q	r	$t = t_0 - \tau$	$t = t_0 + \tau$	$t = 100\tau$
1.0	1.0	1.24×10^{-4}	1.30×10^{-4}	2.79×10^{-4}
1.0	10.0	1.24×10^{-3}	1.30×10^{-3}	3.09×10^{-3}
100.0	1.0	1.24×10^{-4}	1.30×10^{-4}	3.10×10^{-4}
100.0	10.0	1.24×10^{-3}	1.43×10^{-3}	3.16×10^{-3}
10000.0	1.0	1.24×10^{-4}	1.30×10^{-4}	3.10×10^{-4}
10000.0	10.0	1.24×10^{-3}	1.40×10^{-2}	1.58×10^{-2}

The large heat source $q = 10^4$ is probably not physically realistic (such a large input of heat would probably melt the heat resistor!). However, in the test runs such a case shows up the capability of the difference scheme to represent accurately a solution, even when a very large jump in the differential equation was present. It took a large value of r, namely $r = 10.0$, to show up any substantial increase in the error; this being a factor of 10 worse than before the discontinuity. No increase, however, took place after the jump, as can be seen from Table III.

V. (c) A Discontinuous Heat Source Problem Without a Theoretical Solution

We required to study the behaviour of the thermal print head under conditions similar to those experienced in practice; namely, starting initially with a constant (room) temperature and the heat source instantaneously switched on at $t = 0$. We then left the heat source on for a time, studied the build up of temperature in the thin film, and the (slower) build up in the glass substrate. The heat source was then switched off and again the heat dissipation was studied. The results of the computation carried out for a thousand time steps are given in Fig. 3. u_∞ was taken as zero and hence the initial temperature distribution in the print head was also zero. The boundary condition $u(x, y, \ell, t)$ was given to be zero. The graphs represent the temperature at a fixed point ($x = y = \ell/2$) for all time $0 \leqslant t \leqslant 1000\tau$ at several values of z.

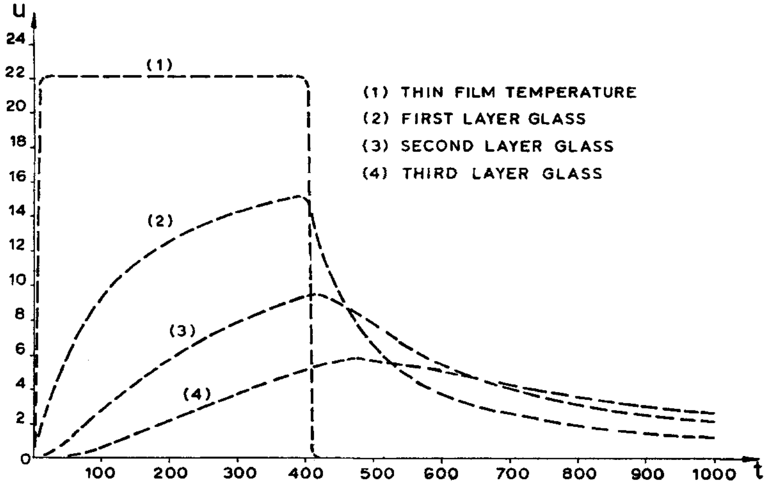

FIG. 3. Graphs of temperature distribution in thermal print head for a single switch-off of the heat source.

V. (d) MULTIPLE SWITCHINGS OF THE HEAT SOURCE

The next set of experiments to be carried out comprised multiple switchings (on and off) of the heat source. In this way, we endeavoured to study the build up, or otherwise, of the temperature in the glass substrate as a function of the printing cycle. The results of the computations are presented in Figs. 4, 5, and 6, where the

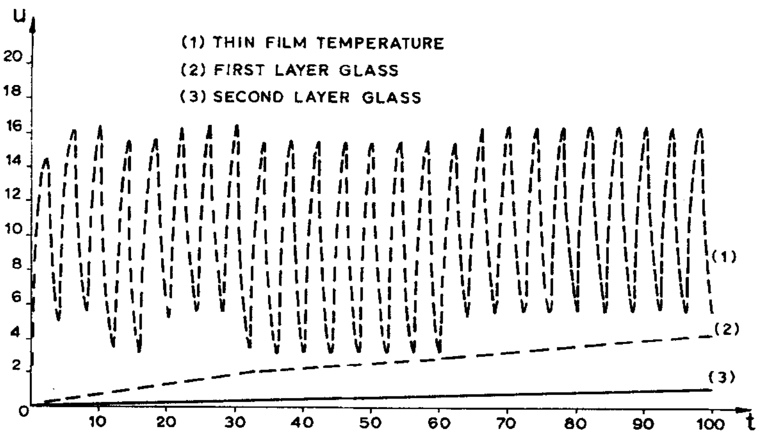

FIG. 4. Graphs of temperature distribution in thermal print head for a printing cycle of 4τ; the heat source being on for 2τ and off for 2τ.

printing cycles were 4, 10, and 20τ, respectively, the cycle being divided equally between "on-time" and "off-time". Figure 7 shows the results for a 20τ cycle where the on-time was 5τ and the off-time 15τ.

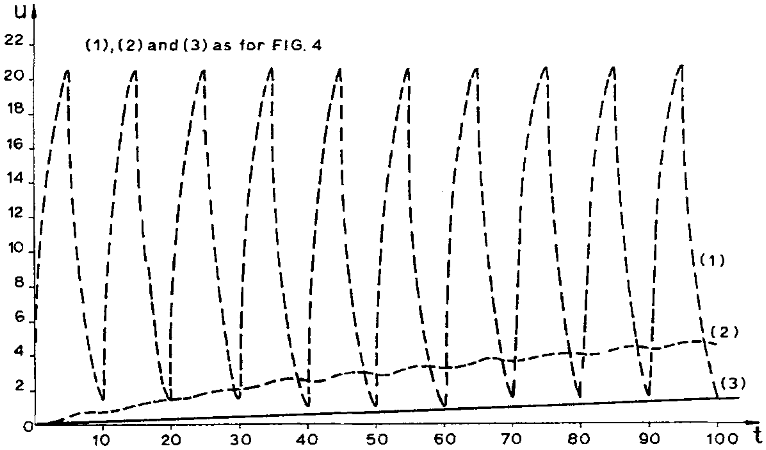

FIG. 5. Graphs of temperature distribution in thermal print head for a printing cycle of 10τ; the heat source being on for 5τ and off for 5τ.

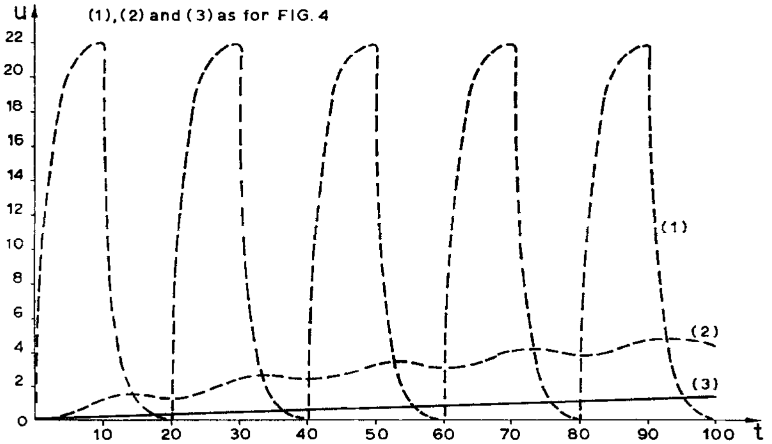

FIG. 6. Graphs of temperature distribution in thermal print head for a printing cycle of 20τ; the heat source being on for 10τ and off for 10τ.

The general rise in temperature in the glass substrate for increasing time is apparent from Figs. 4, 5, and 6. By reducing the "on-time" in comparison with the "off-time," it can be seen from Fig. 7 that the temperature rise in the substrate is [redacted] prevent) the build up of temperature in the glass substrate.

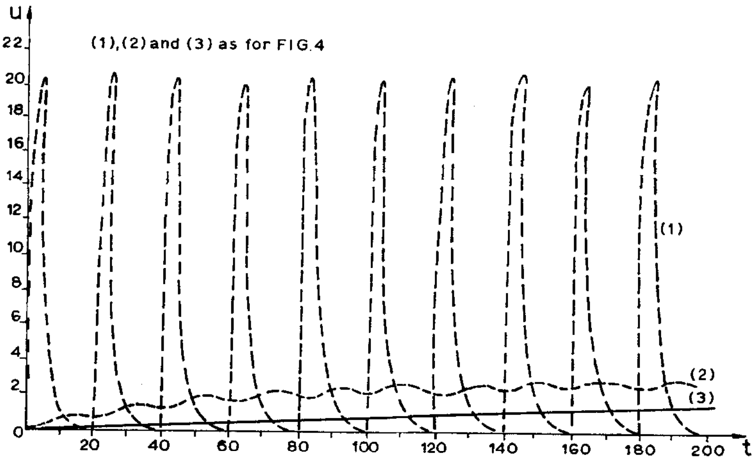

FIG. 7. Graphs of temperature distribution in thermal print head for a printing cycle of 20τ; the heat source being on for 5τ and off for 15τ.

It is interesting to note the slight fluctuation in the thin film temperature (which we will denote by u_1 for the present purpose of identification), as shown in Figs. 4 and 5. There is a correlation between u_1 and the temperature of the uppermost layer in the glass substrate (which we call u_2). For example, in Fig. 4, there is a marked change in the oscillation of u_1 at 30 time steps corresponding to u_2, equal to 2.0. Another marked change in u_1 occurs at 65 time steps, when u_2 equals 3.0. A similar change occurs in u_1 in Fig. 5. However, the fluctuations are not so apparent in Figs. 6 and 7.

In Figs. 4–7, we have indicated the rise in temperature of the second layer in the glass substrate by a straight line; this was because the increase was so slight that fluctuations were not representable on the present scale.

Similar computations to those carried out in Section V were performed for $a \neq 0$ and $2b \neq \ell$, namely for the case where the heating resistor was defined to be smaller than the surface $0 \leqslant x, y \leqslant \ell$, $z = 0$. The results were similar to those described in Figs. 3–7; the main difference being the variation of temperature with x and y at any given time in comparison with the examples described, where, for a particular layer, at any time, the solution was constant over that layer.

VI. Concluding Remarks

It can be concluded from Tables I–III that no error growth or serious loss of accuracy occurs as a result of the discontinuous heat source in Eq. (1). The rapid response of the thin film to switching of the heat source is evident from Figs. 3–7, and the slower response of the glass substrate is also noticeable. It can be seen that for a *given* set of physical coefficients, a printing cycle can be determined in which the surface temperature of the print head can be raised above the threshold temperature of the heat sensitive paper, and in which the build up of substrate temperature can be controlled. If this printing cycle is not satisfactory in practice, then the components of the print head can be changed accordingly, that is, for example, a substrate with a higher conductivity coefficient be used.

The equivalence of the A.D.I. and L.O.D. schemes discussed in Sections III and IV and confirmed by the results of Tables I and II means that the choice of scheme used is decided on the grounds of computational efficiency. Programming-wise, the A.D.I. schemes are logically simpler to use. However, the first step of the D'Yakonov splitting requires a considerable amount of arithmetic manipulation. This disadvantage is alleviated to a certain extent by Gourlay and Mitchell's scheme [14] but, in choosing a particular A.D.I. scheme for this problem, we felt the D'Yakonov splitting was easier to programme. The L.O.D. scheme, however, is both efficient in computer time and is only a little more complicated to programme, owing to the transformations. The time taken to compute the results, using the L.O.D. schemes, is less than the corresponding time by the A.D.I. schemes, provided the solution is not printed every time step; in this situation, the need to alternately transform the u space to the v space and back again adds significantly to the time.

It has become apparent, as a result of this study, that a problem can be considered in two equivalent ways which give rise to two different computational techniques; we refer to the local and global representations of difference schemes. For the linear constant coefficient partial differential equation posed on a rectangular region, the difficulty of noncommuting operators for the local scheme does not arise. However, in order that the schemes do not lose accuracy for time dependent boundary conditions, boundary corrections have to be applied. On the other hand, the global schemes, owing to the incorporation of boundary conditions, can give rise to operators which do not commute. In this case, however, the inclusion of the boundary conditions causes no extra trouble and hence there is no need for boundary corrections. Which scheme to use, in general, cannot be determined here. The particular user must consider which approach is appropriate to his own needs. We chose the global approach because sufficient of the operators commuted in order that the L.O.D. scheme in global form could be applied.

One further point that requires emphasizing is the following. The argument of the vector $\mathbf{g}_{m+1,m}$ contains nodal values in the u space, some of these values coming from the solution of Eq. (1). In this respect, it is imperative that the solution derived for Eq. (1) be in the u space. In our problem, this was, in fact, so; but should it not be so, in a case where the operators do not commute, the transformations would have to be applied at each time step in order to generate the solution in the u space for $\mathbf{g}_{m+1,m}$.

In all the schemes considered, an accuracy of $O(h^2 + \tau^2)$ has been achieved. High accuracy schemes of $O(h^4 + \tau^2)$ could also be proposed; their form is obvious and, hence, will not be given. In agreement with the work of Bramble and Hubbard, in particular [2], it was found that, for the problem posed with mixed function-derivative boundary conditions, no increase in accuracy using the $O(h^4 + \tau^2)$ schemes was attained. In order to obtain the higher accuracy, the derivative boundary conditions would have to be replaced by at least an $O(h^3)$ accurate scheme. In doing this, however, the band structure of the matrices would be destroyed and the operators would no longer commute. Hence, for this particular problem, the "High Accuracy" methods are of no advantage.

In all the numerical experiments, N was taken equal to 10 so that the print head was represented by a cube comprising 1000 grid points. The computation was carried out on an Elliott 4130 computer which contains a 64K word store. The programmes for this particular value of N took up most of the central store. The problem we were restricted to considering, owing to the large storage requirements, was the single element of the 5×5 matrix. The author claims that the results obtained for this problem give insight into the problem of solving the temperature distribution for the whole thermal print head. The justification for this is that in the 5×5 matrix, the normal derivative boundary conditions imposed at the vertical edges of the print head are again insulation boundary conditions. The imposed boundary conditions at the edges of each element, of course, no longer exist, so that the problem is solved for the whole print head with the thin film boundary condition derived from the solution of the differential equations representing the separate elements; the boundary conditions and discontinuities being of a similar nature to those considered in this paper.

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